Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

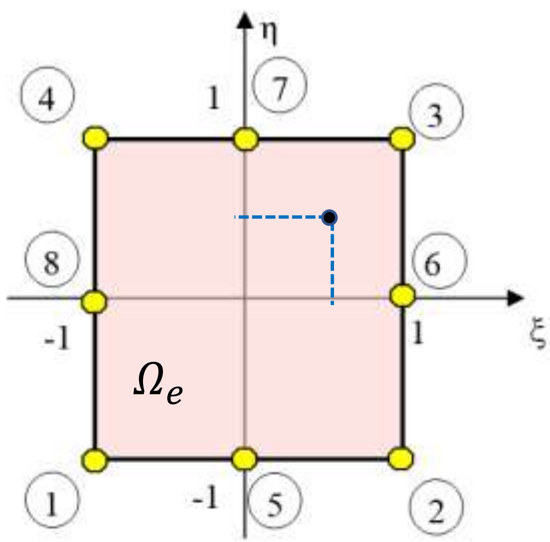
Lecture 4A. 8-node quadrilateral element

03.2025

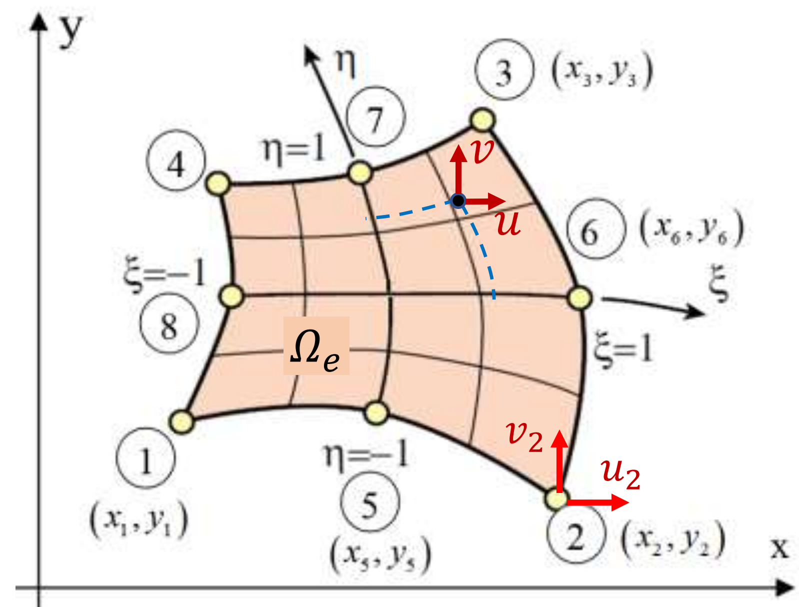
8-node 2D quadrilateral element (accuracy, irregular shapes)

An 8-node element is defined by eight nodes with two degrees of freedom at each node: u_i, v_i. It provides more accurate results and can accommodate irregular shapes without significant loss of accuracy.

natural coordinate system



cartesian coordinate system



$n = 8 \;\; ; \;\; n_p = 2 \;\; \rightarrow \;\; n_e = n \cdot n_p = 16$

Geometry mapping: $(\xi, \eta) \rightarrow (x, y)$

$(-1,-1) \rightarrow (x_1, y_1)$ $\quad (1,-1) \rightarrow (x_2, y_2)$ $\quad (1,1) \rightarrow (x_3, y_3)$ $\quad (-1,1) \rightarrow (x_4, y_4)$

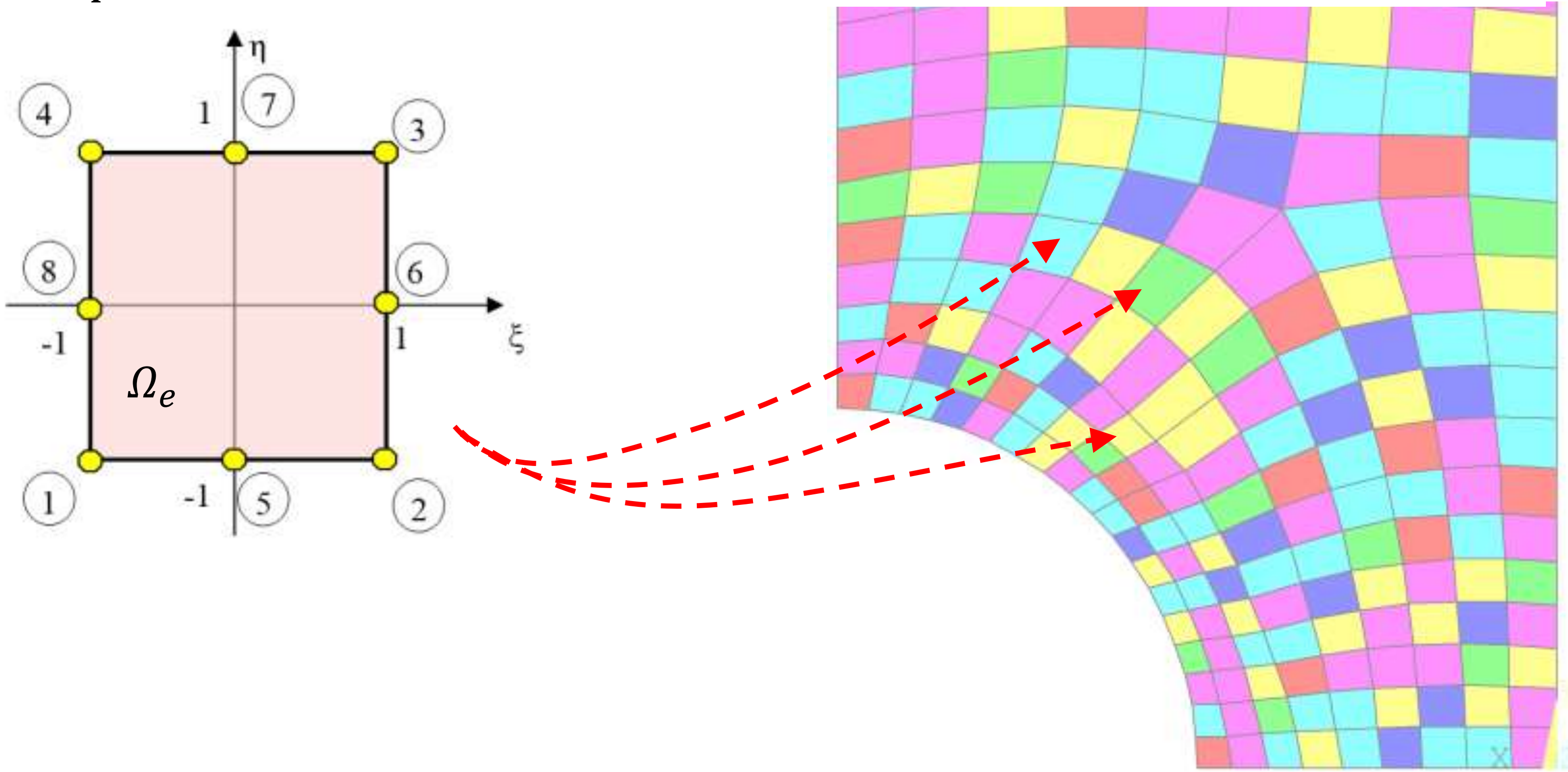
$(0,-1) \rightarrow (x_5, y_5)$ $\quad (1,0) \rightarrow (x_6, y_6)$ $\quad (0,1) \rightarrow (x_7, y_7)$ $\quad (-1,0) \rightarrow (x_8, y_8)$

Reference element vs real elements

One reference element maps to each real element of the mesh

parent element

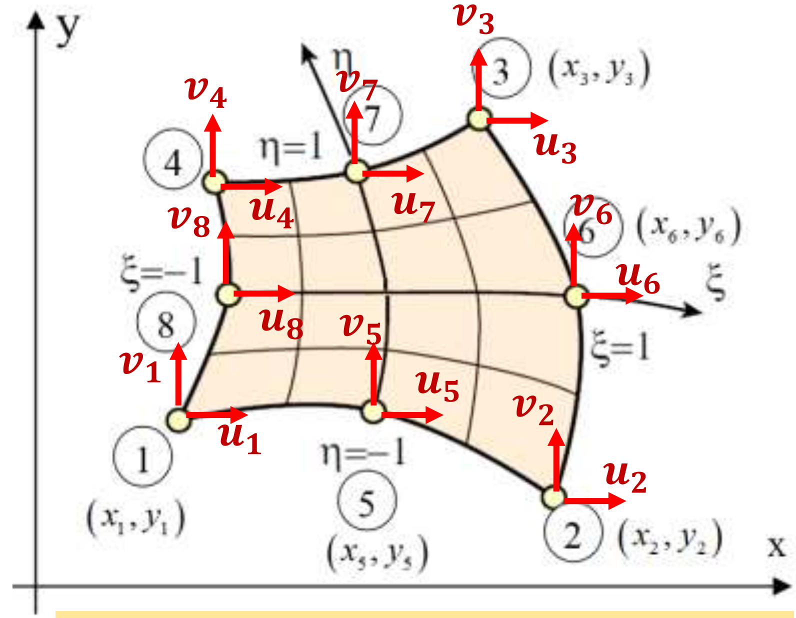
mesh of model elements



Isoparametric mapping

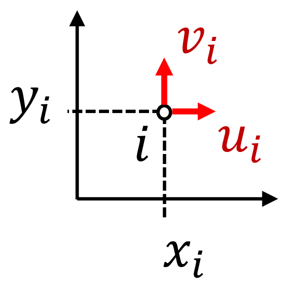
vectors of nodal coordinates

$$\underset{8\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_8 \end{Bmatrix} \quad ; \quad \underset{8\times 1}{\{y_i\}_e} = \begin{Bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_8 \end{Bmatrix}$$



$n = 8 \quad ; \quad n_p = 2 \quad \rightarrow \quad n_e = n \cdot n_p = 16$

local vector of nodal parameters



$$\underset{16\times 1}{\{q\}_e} = \begin{Bmatrix} q_1 \\ q_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ q_{16} \end{Bmatrix}_e = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ . \\ . \\ . \\ u_8 \\ v_8 \end{Bmatrix}_e$$

Isoparametric mapping – the same functions are used to describe the geometry and displacement field

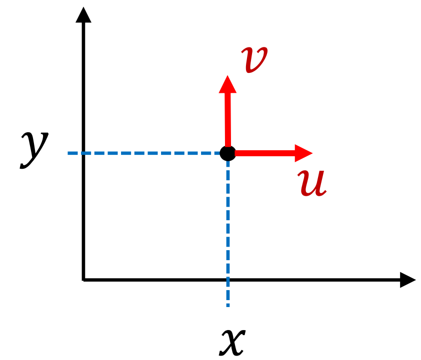
Isoparametric mapping

matrix of shape functions:

$$\underset{2\times 16}{[N(\xi,\eta)]} = \begin{bmatrix} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & \dots & N_8(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & \dots & 0 & N_8(\xi,\eta) \end{bmatrix}$$

vector of shape functions:

$$\underset{1\times 8}{\lfloor N(\xi,\eta)\rfloor} = \lfloor N_1(\xi,\eta), N_2(\xi,\eta), \dots, N_8\,(\xi,\eta)\rfloor$$



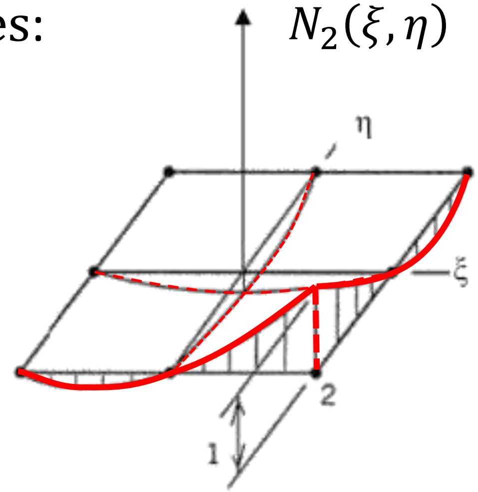
position and displacement of any point:

$$x = \underset{1\times 8}{\lfloor N(\xi,\eta)\rfloor}\underset{8\times 1}{\{x_i\}_e} \quad ; \quad y = \underset{1\times 8}{\lfloor N(\xi,\eta)\rfloor}\underset{8\times 1}{\{y_i\}_e}$$

$$\underset{2\times 1}{\{u\}} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \underset{2\times 16}{[N(\xi,\eta)]}\underset{16\times 1}{\{q\}_e}$$

Shape functions of the 8-node quadrilateral finite elment

corner nodes:



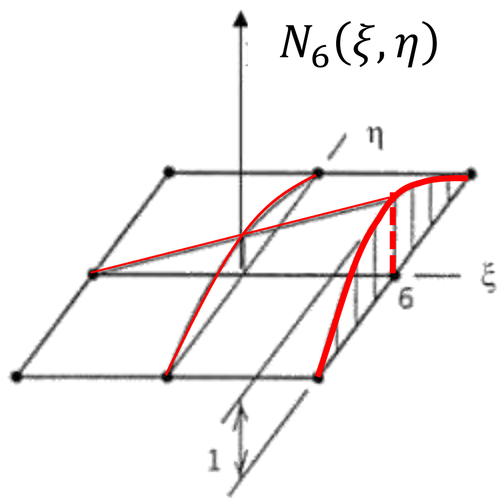
$$N_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_2(\xi,\eta) = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_3(\xi,\eta) = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_4(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

midside nodes:



$$N_5(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_6(\xi,\eta) = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_7(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_8(\xi,\eta) = \frac{1}{2}(1-\xi)(1-\eta^2)$$

Transformation between natural and cartesian coordinates

partial derivatives of any function of coordinates (x, y) with respect to (ξ, η):

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Rightarrow \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \underset{2\times 2}{[J]} \cdot \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix}$$

Jacobian matrix

partial derivatives of any function of coordinates (ξ, η) with respect to (x, y):

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\Rightarrow \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \underset{2\times 2}{[J]^{-1}} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix}$$

inverse Jacobian matrix

Transformation between natural and cartesian coordinates

differential operators:

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \underset{2\times 2}{[J]} \cdot \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \quad ; \quad \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \underset{2\times 2}{[J]}^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

Jacobian matrix — *inverse Jacobian matrix*

differential operators:

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \underset{2\times 2}{[J]} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \underset{2\times 2}{[J]} \cdot \underset{2\times 2}{[J]}^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \underset{2\times 2}{[I]} \cdot \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

unit matrix

How to find the inverse of the Jakobian matrix?

$$[J]=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}$$

the inverse of the Jakobian matrix:

$$\underset{2\times 2}{[J]^{-1}} = \frac{1}{det[J]}\underset{2\times 2}{([J]^{C})^{T}} = \frac{1}{det[J]}\left(\begin{bmatrix}\frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta}\\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi}\end{bmatrix}\right)^{T} = \frac{1}{det[J]}\begin{bmatrix}\frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi}\\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi}\end{bmatrix} =$$

transposed J's cofactor matrix

$$= \begin{bmatrix}\frac{1}{det[J]}\frac{\partial y}{\partial \eta} & -\frac{1}{det[J]}\frac{\partial y}{\partial \xi}\\ -\frac{1}{det[J]}\frac{\partial x}{\partial \eta} & \frac{1}{det[J]}\frac{\partial x}{\partial \xi}\end{bmatrix}$$

$$\underset{2\times 2}{[J]^{-1}} = \begin{bmatrix}\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x}\\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y}\end{bmatrix} \rightarrow$$

geometry approximation *nodal coordinates*

$$\frac{\partial \xi}{\partial x} = \frac{1}{det[J]}\frac{\partial y}{\partial \eta} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} = \frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times 1}{\{y_i\}_e}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{det[J]}\frac{\partial y}{\partial \xi} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi} = -\frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times 1}{\{y_i\}_e}$$

$$\frac{\partial \xi}{\partial y} = -\frac{1}{det[J]}\frac{\partial x}{\partial \eta} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} = -\frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times 1}{\{x_i\}_e}$$

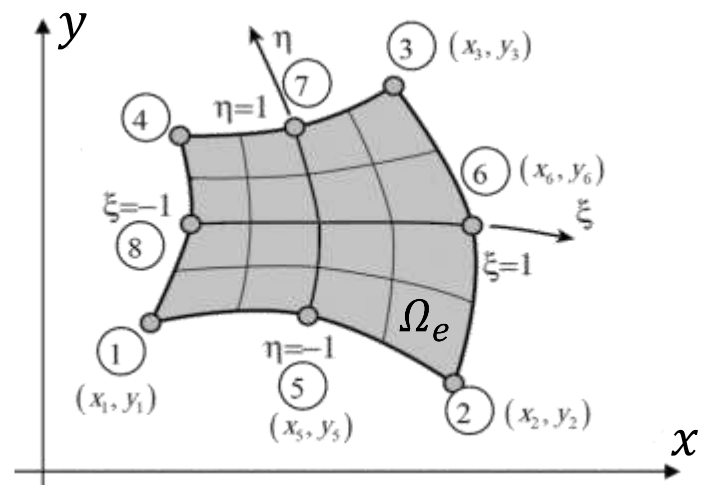
$$\frac{\partial \eta}{\partial y} = \frac{1}{det[J]}\frac{\partial x}{\partial \xi} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi} = \frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times 1}{\{x_i\}_e}$$

How to find the inverse of the Jakobian matrix?

determinant of the Jakobian matrix:

geometry approximation

$$det[J] = det\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} =$$



$$= \frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} - \frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} =$$

$$= \left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times 1}{\{x_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times 1}{\{y_i\}_e}\right) - \left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times 1}{\{y_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times 1}{\{x_i\}_e}\right)$$

(known at any point of the domain Ω_e)

Gradient matrix calculation

differential operators in the coordinate system (x, y):

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \underset{2\times 2}{[J]^{-1}} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{1}{det[J]}\frac{\partial y}{\partial \eta} & -\frac{1}{det[J]}\frac{\partial y}{\partial \xi} \\ -\frac{1}{det[J]}\frac{\partial x}{\partial \eta} & \frac{1}{det[J]}\frac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} \rightarrow$$

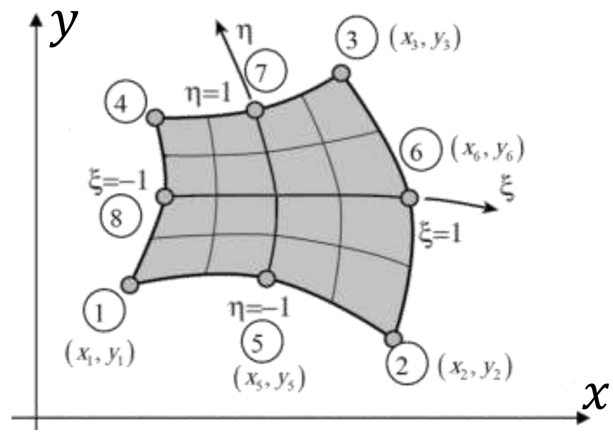
$$\frac{\partial}{\partial x} = \frac{1}{det[J]}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{1}{det[J]}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta} \quad ; \quad \frac{\partial}{\partial y} = -\frac{1}{det[J]}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi} + \frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}$$

gradient matrix for plane stress or plane strain conditions:

$$\underset{3\times 2}{[R]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{1}{det[J]}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right) & 0 \\ 0 & \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{1}{det[J]}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right) \\ \left(\frac{1}{det[J]}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{1}{det[J]}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right) & \left(\frac{1}{det[J]}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{1}{det[J]}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right) \end{bmatrix} = \underset{3\times 2}{[R(\xi, \eta)]}$$

Gradient matrix for plane stress and plane strain conditions

$$\underset{3\times 2}{[R(\xi,\eta)]} = \begin{bmatrix} \frac{\partial}{\partial x}(\xi,\eta) & 0 \\ 0 & \frac{\partial}{\partial y}(\xi,\eta) \\ \frac{\partial}{\partial y}(\xi,\eta) & \frac{\partial}{\partial x}(\xi,\eta) \end{bmatrix}$$



$$\underset{8\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{Bmatrix} \qquad \underset{8\times 1}{\{y_i\}_e} = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_8 \end{Bmatrix}$$

$$\frac{\partial}{\partial x}(\xi,\eta) = \frac{\left(\overset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\overset{8\times 1}{\{y_i\}_e}\right)\frac{\partial}{\partial\xi} - \left(\overset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\overset{8\times 1}{\{y_i\}_e}\right)\frac{\partial}{\partial\eta}}{\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\underset{8\times 1}{\{x_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\underset{8\times 1}{\{y_i\}_e}\right) - \left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\underset{8\times 1}{\{y_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\underset{8\times 1}{\{x_i\}_e}\right)}$$

$$\frac{\partial}{\partial y}(\xi,\eta) = \frac{\left(\overset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\overset{8\times 1}{\{x_i\}_e}\right)\frac{\partial}{\partial\eta} - \left(\overset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\overset{8\times 1}{\{x_i\}_e}\right)\frac{\partial}{\partial\xi}}{\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\underset{8\times 1}{\{x_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\underset{8\times 1}{\{y_i\}_e}\right) - \left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\xi}}\underset{8\times 1}{\{y_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial\eta}}\underset{8\times 1}{\{x_i\}_e}\right)}$$

Strain energy of an 8-node QUAD element

strain vector for plane stress or plane strain conditions:

$$\underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 2}{[R(\xi,\eta)]}\underset{2\times 1}{\{u\}} = \underset{3\times 2}{[R(\xi,\eta)]}\underset{2\times 16}{[N(\xi,\eta)]}\underset{16\times 1}{\{q\}_e} = \underset{3\times 16}{[B(\xi,\eta)]}\underset{16\times 1}{\{q\}_e}$$

elastic strain energy of a finite element:

$$U_e = \frac{1}{2}\int_{\Omega_e} \underset{1\times 3}{\lfloor\varepsilon\rfloor}\underset{3\times 1}{\{\sigma\}}\, d\Omega_e = \frac{1}{2}\underset{1\times 16}{\lfloor q\rfloor_e}\left(t_e \int_{A_e} \underset{16\times 3}{[B(\xi,\eta)]^T}\underset{3\times 3}{[D]}\underset{3\times 16}{[B(\xi,\eta)]}\, dxdy\right)\underset{16\times 1}{\{q\}_e} =$$

$$\underset{3\times 1}{\{\sigma\}} = \underset{3\times 3}{[D]}\underset{3\times 1}{\{\varepsilon\}}$$

$$= \frac{1}{2}\lfloor q\rfloor_e [k]_e \{q\}_e$$

$$\underset{1\times 3}{\lfloor\varepsilon\rfloor} = \underset{1\times 16}{\lfloor q\rfloor_e}\underset{16\times 3}{[B(\xi,\eta)]^T} \qquad \underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 16}{[B(\xi,\eta)]}\underset{16\times 1}{\{q\}_e}$$

$$\left(\int_{A_e} f(x,y)\, dxdy = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\det[J]\, d\xi d\eta\right)$$

local stiffness matrix:

$$\underset{16\times 16}{[k]_e} = t_e \int_{-1}^{1}\int_{-1}^{1} \underset{16\times 3}{[B(\xi,\eta)]^T}\underset{3\times 3}{[D]}\underset{3\times 16}{[B(\xi,\eta)]}\det[J]\, d\xi d\eta$$

(calculated numerically)

Potential energy of loading and equivalent load vector

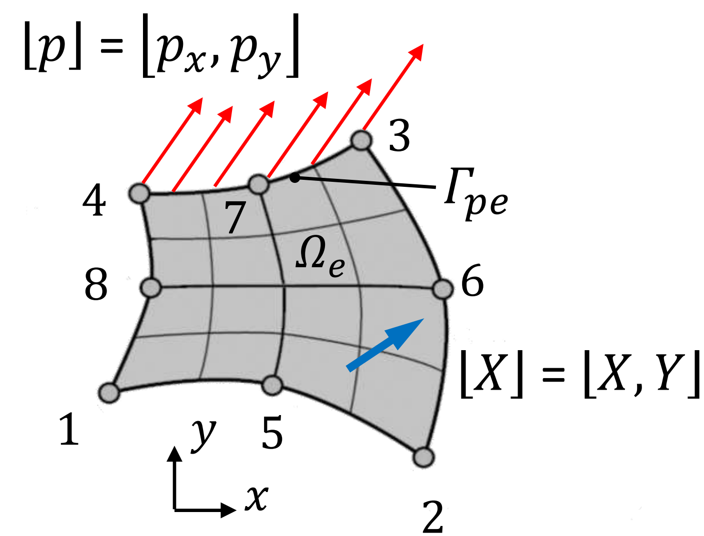
potential energy of loading in a finite element:

$$W_e = \int_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 1}{\{u\}} \, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 1}{\{u\}} \, d\Gamma_{pe} =$$

$$\underset{2\times 1}{\{u\}} = \underset{2\times 16}{[N]} \underset{16\times 1}{\{q\}_e}$$

$$= \left(\int_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor} \underset{2\times 16}{[N]} \, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} \, d\Gamma_{pe} \right) \underset{16\times 1}{\{q\}_e} =$$

$$= \left(\underset{1\times 16}{\lfloor F^X \rfloor_e} + \underset{1\times 16}{\lfloor F^p \rfloor_e} \right) \underset{16\times 1}{\{q\}_e} = \underset{1\times 16}{\lfloor F \rfloor_e} \underset{16\times 1}{\{q\}_e}$$

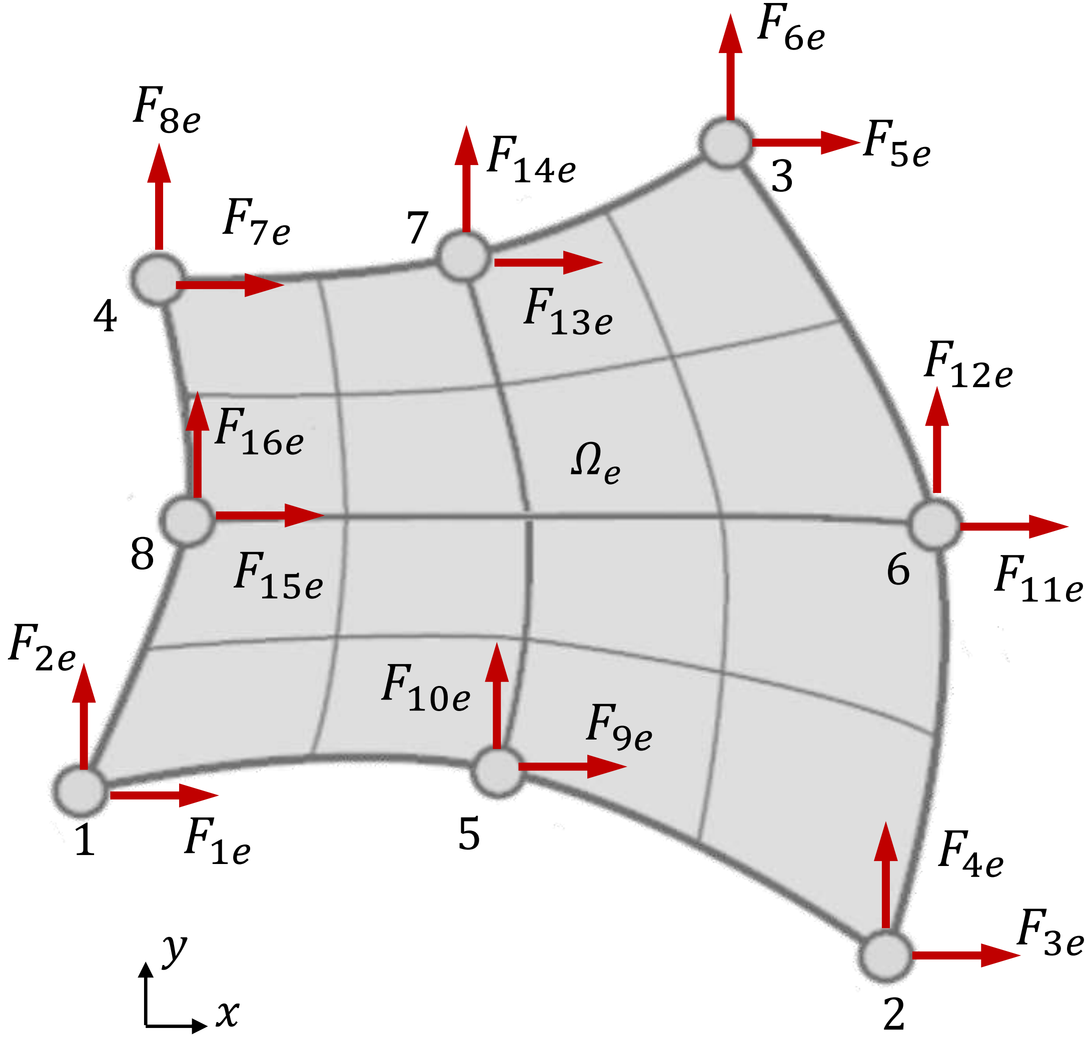


$$\underset{1\times 16}{\lfloor F^X \rfloor_e} = t_e \int_{-1}^{1} \int_{-1}^{1} \underset{1\times 2}{\lfloor X(\xi,\eta) \rfloor} \underset{2\times 16}{[N(\xi,\eta)]} \, det[J] \, d\xi d\eta$$

$$\underset{1\times 16}{\lfloor F^p \rfloor_e} = \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} d\Gamma_{pe}$$

Equivalent load vector in the 8-node quadrilateral element

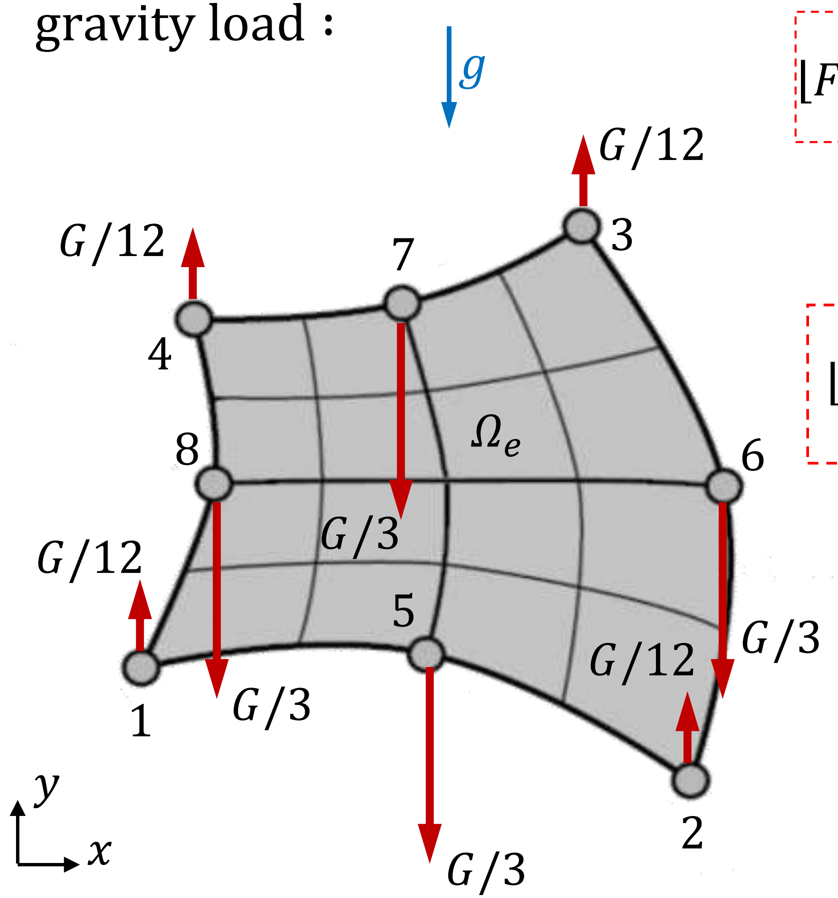
$\underset{16\times 1}{\lfloor F \rfloor_e}$



$$\{F\}_e = \begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F_{16} \end{Bmatrix}_e$$

Example. Equivalent load vector due to mass forces *(gravity load)*

gravity load :



$$\lfloor F^X \rfloor_e = t_e \int_{-1}^{1}\int_{-1}^{1} \lfloor X(\xi,\eta) \rfloor [N(\xi,\eta)]\; det[J]\; d\xi d\eta$$

$$\underset{1\times 2}{\lfloor X \rfloor}$$

$$\underset{1\times 16}{\lfloor F^X \rfloor_e} = t_e \int_{-1}^{1}\int_{-1}^{1} \lfloor 0, -\rho g \rfloor \underset{2\times 16}{[N(\xi,\eta)]}\; det[J]\; d\xi d\eta$$

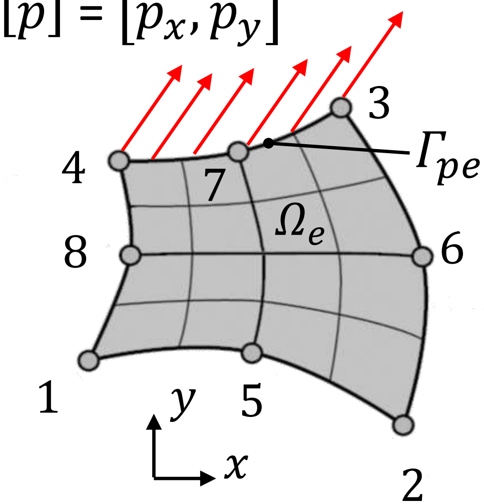
$$G = \rho g \Omega_e \; (N)$$

$$\underset{1\times 16}{\lfloor F \rfloor_e} = \left\lfloor 0, \frac{G}{12}, 0, \frac{G}{12}, 0, \frac{G}{12}, 0, \frac{G}{12}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, \right\rfloor_e$$

Equivalent load vector due to surface load

equivalent load vector due to surface load:

$$\lfloor p \rfloor = \lfloor p_x, p_y \rfloor$$



$$\underset{1\times 16}{\lfloor F^p \rfloor_e} = \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} d\Gamma_{pe} = t_e \int_0^l \lfloor p \rfloor [N] ds$$

$$= t_e \int_0^l \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} ds = t_e \int_{-1}^1 \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} \frac{ds}{d\xi} d\xi$$

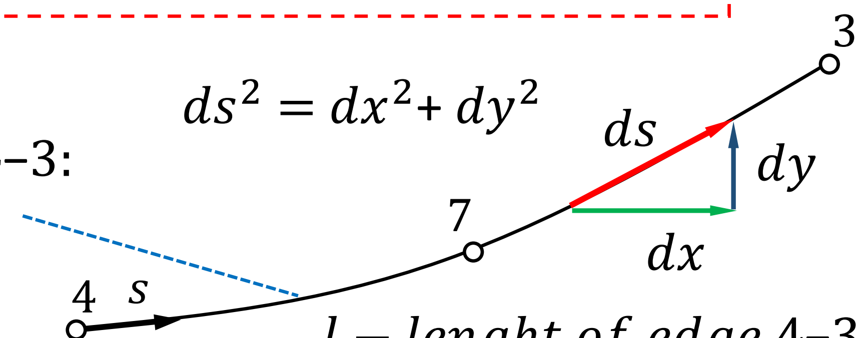
$$\frac{ds^2}{d\xi^2} = \frac{dx^2}{d\xi^2} + \frac{dy^2}{d\xi^2} \quad \rightarrow \frac{ds}{d\xi} = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$$

$$\underset{1\times 16}{\lfloor F^p \rfloor_e} = t_e \int_{-1}^1 \lfloor p_x, p_y \rfloor \underset{2\times 16}{[N]} \sqrt{\left(\underset{1\times 8}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \underset{8\times 1}{\{x_i\}_e}\right)^2 + \left(\underset{1\times 8}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \underset{8\times 1}{\{y_i\}_e}\right)^2}\, d\xi$$

(calculated numerically)

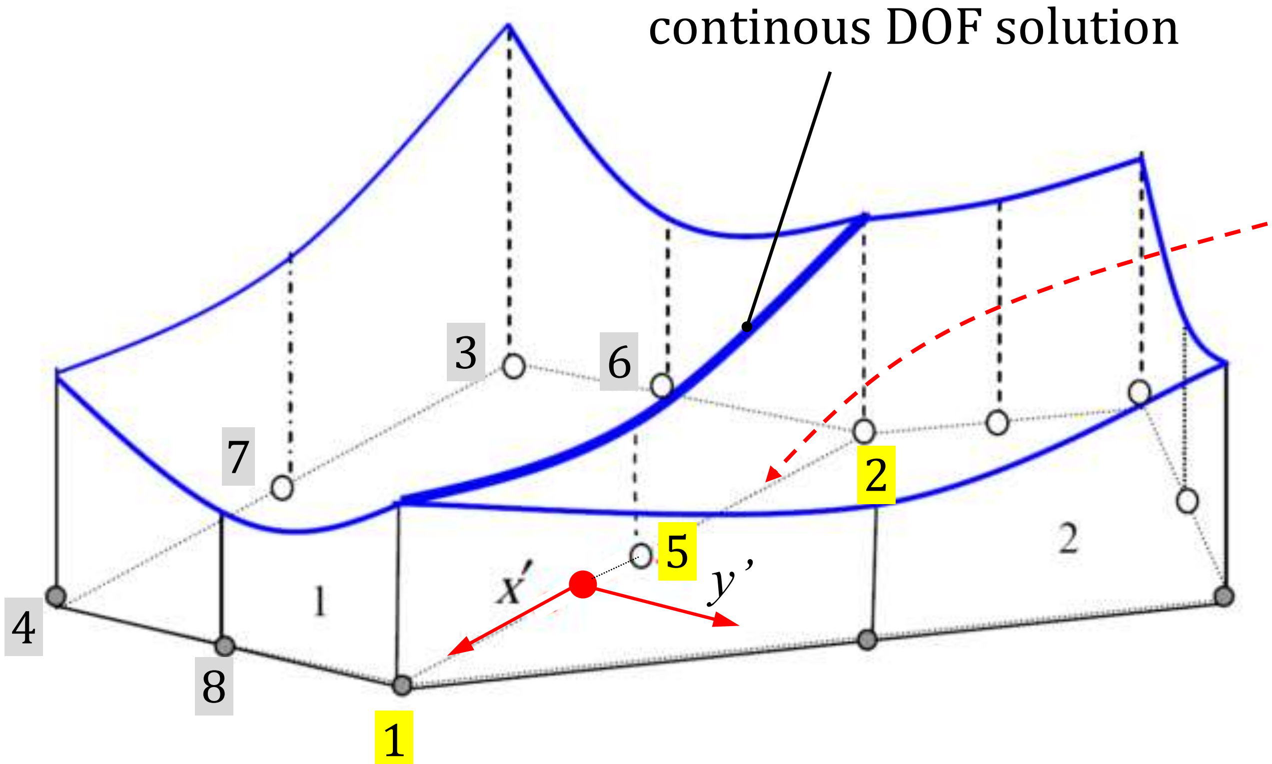
$$ds^2 = dx^2 + dy^2$$

on edge 4–3:
$\eta = 1$



l – lenght of edge 4–3

Results on the boundary between two 8-node FEs



$$x = (x_1, x_5, x_2)$$
$$y = (y_1, y_5, y_2)$$

$$u = (u_1, u_5, u_2)$$
$$v = (v_1, v_5, v_2)$$

$$\left.\frac{\partial u}{\partial x'}\right|_1 = \left.\frac{\partial u}{\partial x'}\right|_2 \qquad \left.\frac{\partial u}{\partial y'}\right|_1 \neq \left.\frac{\partial u}{\partial y'}\right|_2 \quad \Rightarrow \quad (\varepsilon_{x'})_1 = (\varepsilon_{x'})_2 \qquad (\varepsilon_{y'})_1 \neq (\varepsilon_{y'})_2 \quad \Rightarrow (\sigma_{ij})_1 \neq (\sigma_{ij})_2$$

not continous element solution

Example. 2D model of cantilever beam *(8-node FEs)*